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On Nevai's Bounds for Orthogonal Polynomials Associated with Exponential Weights

D. S. LUBINSKY

National Research Institute for Mathematical Sciences of the CSIR, P.O. Box 395, Pretoria 0001, Republic of South Africa

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Using ideas of Freud (*J. Approx. Theory* **19** (1977), 22–37) Mhaskar and Saff (*Trans. Amer. Math. Soc.* **285** (1984), 203–234, and Nevai (*J. Approx. Theory* **44**, No. 1 (1985)), we obtain bounds for $p_n(x)-p_{n-2}(x)$ and related expressions, for all $x\in\mathbb{R}$, where $p_n(x)$ is the orthonormal polynomial of degree n for the weight $\exp(-x^m)$, m a positive even integer. © 1985 Academic Press, Inc.

1. STATEMENT OF RESULTS

Let $w(x)=\exp(-x^m)$, $x\in\mathbb{R}$, where m is a fixed positive even integer. Let $\{p_n\}_{n=0}^{\infty}$ denote the corresponding system of orthonormal polynomials. Let $\{a_n\}_{n=0}^{\infty}$ be the coefficients in the recurrence relation

$$xp_n=a_{n+1}p_{n+1}+a_np_{n-1}, \qquad n=1,2,3\ldots .$$

In this note, we use estimates of Nevai from [11], two inequalities of Freud from [4], and an identity of Mhaskar and Saff [9] to prove the following result. Throughout, $C, C_1, C_2,\ldots,$ denote positive constants independent of n and x.

THEOREM A.

(i) $w(x)(p_n(x)-p_{n-2}(x))^2\leqslant Cn^{-1/m}$, $x\in\mathbb{R}$. (1)

(ii) $w(x)p_n^2(x)|1-(x/(2a_n))^2|\leqslant Cn^{-1/m}$, $x\in\mathbb{R}$. (2)

(iii) *Given $\varepsilon>0$, there exists $C=C(\varepsilon)$ such that*

$$w(x)p_n^2(x)\leqslant C, \tag{3}$$

for all $x\in\mathbb{R}$ with $|2a_n-|x||\geqslant\varepsilon$.

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Remarks. (a) Results of Bonan [2] and Nevai [11] imply (1), (2), and (3) for $|x| \leqslant Cn^{1/m}$ and certain choices of C. Nevai [10, p. 193] conjectured that (3) is true for all $x \in \mathbb{R}$.

(b) One may replace a_n in (2) by a_{n+k}, where k is any fixed integer.

(c) The proof uses, first, the identity (see Dombrowski and Fricke [3] or see (6) in [11])

$$\sum_{k=0}^{n-1} (a_{k+1}^2 - a_k^2)\, p_k^2(x)$$
$$= a_n^2(p_{n-1}(x) - xp_n(x)/(2a_n))^2 + a_n^2(1 - x^2/(4a_n^2))\, p_n^2(x)$$
$$= a_n^2\{p_{n-1}^2(x) - xp_n(x)\, p_{n-1}(x)/a_n + p_n^2(x)\},$$

which, as in Nevai [11], yields

THEOREM B.

$$w(x)|\, p_{n-1}^2(x) - xp_n(x)\, p_{n-1}(x)/a_n + p_n^2(x)| \leqslant Cn^{-1/m}, \qquad x \in \mathbb{R}. \tag{4}$$

Use is made of asymptotics for a_n due to Magnus [6], Lew and Quarles [5], Mate and Nevai [7] and Mate, Nevai, and Zaslavsky [8]—see (7) in Nevai [11]:

$$2a_n = \beta n^{1/m} + O(n^{1/m-2}), \tag{5}$$

where

$$\beta = \{\pi^{1/2}\Gamma(m/2)/\Gamma((m+1)/2)\}^{1/m}. \tag{6}$$

Finally, we shall also need the following theorem of Mhaskar and Saff [9, Theorem 2.7]:

THEOREM C. *For all polynomials P of degree at most n,*

$$\| Pw \|_{L_\infty(\mathbb{R})} = \| Pw \|_{L_\infty[-a_n^*, a_n^*]},$$

where

$$a_n^* = (n/\lambda_m)^{1/m}, \tag{7}$$

and

$$\lambda_m = 2^{1-m}\Gamma(m+1)/\{\Gamma(m/2)\,\Gamma(m/2+1)\}. \tag{8}$$

The quantity a_n^* is denoted by $a_n(\alpha) = a_n(m)$ in Mhaskar and Saff [5]. Further, λ_m is given by (1.6) in [9].

2. Proofs

The following lemma in a sense states that intervals of length $o(n^{-1+1/m})$ do not matter much for supremum norms of polynomials of degree $\leqslant n$.

Lemma 1. *Let $\{A_n\}$ be a sequence of positive numbers such that*

$$a_n^* - A_n = o(n^{-1+1/m}), \qquad n \to \infty. \tag{9}$$

Let $\{Q_n\}$ be a sequence of polynomials such that Q_n has degree at most n. Let

$$B_n = \| Q_n w \|_{L_\infty(\mathbb{R})}, \qquad n = 1, 2, \ldots, \tag{10}$$

and

$$b_n = \| Q_n w \|_{L_\infty[-A_n, A_n]}, \qquad n = 1, 2, \ldots . \tag{11}$$

Then

$$\lim_{n \to \infty} B_n / b_n = 1. \tag{12}$$

Proof. For those n for which $A_n \geqslant a_n^*$, Theorem C ensures that $B_n = b_n$. Hence we may assume $A_n < a_n^*$, $n = 1, 2, 3, \ldots$. Let $A_n < x \leqslant a_n^*$. There exists $u \in (A_n, x)$ such that

$$\begin{aligned}(Q_n w)(x) &= (Q_n w)(A_n) + (x - A_n)(Q_n w)'(u) \\ &= (Q_n w)(A_n) + (x - A_n)(Q_n'(u)\, w(u) - m u^{m-1} Q_n(u)\, w(u)).\end{aligned} \tag{13}$$

We note that, by Theorem 1.1 in Freud [4, p. 23],

$$|Q_n'(u)\, w(u)| \leqslant C_1 n^{1-1/m} \| Q_n w \|_{L_\infty(\mathbb{R})},$$

while $|u|^{m-1} \leqslant (a_n^*)^{m-1} \leqslant C_2 n^{1-1/m}$. Then (10), (11), and (13) yield for all $x \in (A_n, a_n^*]$,

$$|Q_n w|(x) \leqslant b_n + (a_n^* - A_n)\, C n^{1-1/m} B_n. \tag{14}$$

Similarly, we may deal with $x \in [-a_n^*, -A_n)$. Further, (14) holds trivially for $x \in [-A_n, A_n]$. Then Theorem C yields

$$B_n \leqslant b_n + (a_n^* - A_n)\, C n^{1-1/m} B_n$$

or

$$B_n \leqslant b_n (1 - (a_n^* - A_n)\, C n^{1-1/m})^{-1},$$

so that, by (9),

$$\limsup_{n\to\infty} B_n/b_n \leqslant 1.$$

As $B_n \geqslant b_n$, (12) follows. ▮

We shall apply Lemma 1 with $A_{2n}=2a_n$. To this end, we must establish (9). Write $m=2l$. By (5), (6), (7), and (8),

$$\begin{aligned}\frac{2a_n}{a^*_{2n}} &= \left\{\frac{\pi^{1/2}\Gamma(m/2)\,2^{1-m}\Gamma(m+1)}{\Gamma((m+1)/2)\,2\Gamma(m/2)\,\Gamma(m/2+1)}\right\}^{1/m} + O(n^{-2})\\ &= \left\{\frac{\pi^{1/2}(l-1)!\,2^{-2l}(2l)!}{(l-1/2)(l-3/2)\cdots(1/2)\,\pi^{1/2}\,(l-1)!\,l!}\right\}^{1/m} + O(n^{-2})\\ &= 1+O(n^{-2}),\end{aligned}$$

so that

$$a^*_{2n} - 2a_n = O(n^{1/m-2}), \qquad n\to\infty. \tag{15}$$

Proof of Theorem A

We first establish the following statement: Let k be an integer. Then, for all $x\in\mathbb{R}$,

$$w(x)|\,p_n^2(x) - (x/a_{n+k})\,p_n(x)\,p_{n-1}(x) + p_{n-1}^2(x)| \leqslant Cn^{-1/m}. \tag{16}$$

In fact, this follows from (4), provided we can show that for all $x\in\mathbb{R}$,

$$w(x)|\,xp_n(x)\,p_{n-1}(x)\{1/a_{n+k} - 1/a_n\}| \leqslant Cn^{-1/m}. \tag{17}$$

First, note the rather weak inequality

$$p_n^2(x)\,w(x) \leqslant Cn^{1-1/m}, \qquad x\in\mathbb{R}, \tag{18}$$

which follows from Lemma 2.5 in Freud [4, p. 25] or from inequality (8) in Nevai [11]. Next, by (5),

$$a_{n+k} - a_n = O(n^{1/m-1}), \qquad n\to\infty. \tag{19}$$

From (5), (18), and (19), we see that the left member of (17) is bounded for $|x|\leqslant 2a_n$ by $C_1 n^{-1/m}$. Applying Lemma 1 with $A_{2n}=2a_n$ and $Q_{2n}(x)=xp_n(x)\,p_{n-1}(x)$, and noting that (15) implies (9) for positive even integers, we see that (17), and hence (16), holds for all $x\in\mathbb{R}$.

Proof of (i) *of Theorem A.* We apply (16) with $k=1$. Let $0\leqslant x\leqslant 2a_{n+1}$. If $p_n(x)\,p_{n-1}(x)<0$, (16) shows

$$Cn^{-1/m} \geqslant w(x)(p_n^2(x) + p_{n-1}^2(x))$$
$$\geqslant w(x)(p_n(x) - p_{n-1}(x))^2/2.$$

On the other hand, if $p_n(x)\,p_{n-1}(x) > 0$, (16) shows

$$Cn^{-1/m} \geqslant w(x)(p_n^2(x) - 2p_n(x)\,p_{n-1}(x) + p_{n-1}^2(x))$$
$$= w(x)(p_n(x) - p_{n-1}(x))^2.$$

Hence for $0 \leqslant x \leqslant 2a_{n+1}$,

$$w(x)(p_n(x) - p_{n-1}(x))^2 \leqslant Cn^{-1/m}. \tag{20}$$

By considering (20) for n and $n-1$, we obtain

$$w(x)(p_n(x) - p_{n-2}(x))^2 \leqslant Cn^{-1/m}, \tag{21}$$

$0 \leqslant x \leqslant 2a_n$. As $(p_n(x) - p_{n-2}(x))^2$ is even, it follows that (21) holds for $|x| \leqslant 2a_n$. Applying Lemma 1 with $Q_{2n} = (p_n - p_{n-2})^2$ and $A_{2n} = 2a_n$, we obtain that (21) holds for all $x \in \mathbb{R}$. ∎

Proof of (ii) *of Theorem A*. Applying (16) with $k=1$, we see that for $|x| \leqslant 2a_{n+1}$,

$$p_n^2(x) - (x/a_{n+1})\,p_n(x)\,p_{n-1}(x) + p_{n-1}^2(x)$$
$$= (p_{n-1}(x) - (x/(2a_{n+1}))\,p_n(x))^2 + (1 - (x/(2a_{n+1}))^2)\,p_n^2(x)$$
$$\geqslant (1 - (x/(2a_{n+1}))^2)\,p_n^2(x) \geqslant 0.$$

Then, using (16) and Lemma 1 with $Q_{2n+2} = (1 - (x/(2a_{n+1}))^2)\,p_n^2$ and $A_{2n+2} = 2a_{n+1}$, we obtain

$$w(x)|\,1 - (x/(2a_{n+1}))^2|\,p_n^2(x) \leqslant Cn^{-1/m}, \qquad x \in \mathbb{R}. \tag{22}$$

Using (18) and (19), we may easily prove

$$w(x)(x/2)^2\,p_n^2(x)|\,a_{n+1}^{-2} - a_n^{-2}| \leqslant Cn^{-1/m}, \qquad x \in \mathbb{R}. \tag{23}$$

Then (22) and (23) yield (2). ∎

Proof of (iii) *of Theorem A*. For $|2a_n - |x|| \geqslant \varepsilon$, we have

$$|1 - (x/(2a_n))^2| = |1 - |x|/(2a_n)|\,|1 + |x|/(2a_n)|$$
$$\geqslant \varepsilon/(2a_n) \geqslant C\varepsilon n^{-1/m},$$

and then (2) yields (3). ∎

Remarks. (a) For the Hermite weight ($m=2$), a better inequality than (1) appears in Askey and Wainger [1, p. 700].

(b) After stating Theorem 1 in [11], Nevai conjectures that in this theorem $0<c<1$ cannot be replaced by $c=1$. If (2) in [11] holds with $c=1$, Theorem C shows that

$$w(x)\,p_n^2(x) \leqslant Cn^{-1/m}, \qquad x \in \mathbb{R},$$

but this does not readily lead to a contradiction.

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